

The multiplets of finite-width 0^{++} mesons and encounters with exotics

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Received: 7 October 2005 / Revised version: 22 December 2005 /

Published online: 30 March 2006 – © Springer-Verlag / Società Italiana di Fisica 2006

Abstract. The complex-mass (finite-width) 0^{++} nonet and decuplet are investigated by means of the exotic commutator method. The hypothesis of the vanishing of the exotic commutators leads to the system of master equations (ME). Solvability conditions of these equations define relations between the complex masses of the nonet and decuplet mesons which, in turn, determine relations between the real masses (mass formulae), as well as between the masses and widths of the mesons. Mass formulae are independent of the particle widths. The masses of the nonet and decuplet particles obey simple ordering rules. The nonet mixing angle and the mixing matrix of the isoscalar states of the decuplet are completely determined by solution of ME; they are real and do not depend on the widths. All known scalar mesons with the mass smaller than 2000 MeV (excluding $\sigma(600)$) and one with the mass $2200 \div 2400$ MeV belong to two multiplets: the nonet ($a_0(980)$, $K_0(1430)$, $f_0(980)$, $f_0(1710)$) and the decuplet ($a_0(1450)$, $K_0(1950)$, $f_0(1370)$, $f_0(1500)$, $f_0(2200)/f_0(2330)$). It is shown that the famed anomalies of the $f_0(980)$ and $a_0(980)$ widths arise from an extra “kinematical” mechanism, suppressing decay, which is not conditioned by the flavor coupling constant. Therefore, they do not justify rejecting the $q\bar{q}$ structure of them. A unitary singlet state (glueball) is included into the higher lying multiplet (decuplet) and is divided among the $f_0(1370)$ and $f_0(1500)$ mesons. The glueball contents of these particles are totally determined by the masses of decuplet particles. Mass ordering rules indicate that the meson $\sigma(600)$ does not mix with the nonet particles.

1 Introduction

Thirty years ago David Morgan posed the question of the “respectability” of scalar mesons as $q\bar{q}$ systems [1]. He attempted to find an affirmative answer to this question. Soon after that, people became sceptical about such a possibility. Primarily, the main reason was the supposed domination of the $f_0(980) \rightarrow K\bar{K}$ decay channel. Later, after establishing that this decay is not dominating (PDG have been announcing domination of the mode $f_0(980) \rightarrow \pi\pi$ since 1982), the disagreement between the measured Γ^{exp} [2] and the predicted $\Gamma^{q\bar{q}}$ [3] total width of the decay:

$$\Gamma^{\text{exp}} = 40 \div 100 \text{ MeV}, \quad (1)$$

$$\Gamma^{q\bar{q}} = 500 \div 1000 \text{ MeV} \quad (2)$$

was recognized as a main argument against the $q\bar{q}$ structure of the $f_0(980)$ meson. Probably, this argument was never contested.

So a question arose as to the internal structure of the $f_0(980)$, $a_0(980)$ and other mesons forming a scalar multiplet. Many alternative models were created to explain

this multiplet. The most prominent ones are exotic models describing scalar mesons totally or partly as $qq\bar{q}\bar{q}$ states. These models differ from one another with physical interpretation and/or construct the multiplet from different particles. It is not the purpose of this paper to discuss these issues. In the extensive bibliography introducing these models the interested reader is referred to a few representative papers [4–8]. It should be, however, recognized that the views of many authors evolved remarkably during the time elapsed. We abandon discussing them, because we question the validity of the argument that the disagreement between the numbers of (1) and (2) can be regarded as evidence against the $q\bar{q}$ nature of the scalar mesons. Consequently, we question the exotic models of the scalar nonet.

Although the arguments against rejection of the $q\bar{q}$ model will be set forth later, it is worth noting here that the disagreement between (1) and (2) by itself does not certify a contradiction. The contradiction only appears if one admits that the observed width is determined entirely by the flavor coupling constant. Such a point of view is widely shared, in spite of many examples of hadronic decays revealing additional suppression. The reason is that there is no known mechanism which could suppress the $f_0(980)$ decay. However, as we argue below, such a mechanism must

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exist. Its indispensability is clearly seen if the mesons are described as finite-width nonet states.

Below, we use the notion of the “flavor width” (FW) which is distinguished from the “hadronic width” (HW). It has been shown that the total FW which is determined by the flavor coupling constant is reduced to an experimentally observed total HW due to some “kinematical” suppression mechanism [9]. Thus the number from (1) is the HW, while the number from (2) is the FW of the $f_0(980)$ meson.

Another kind of exotics is being searched. According to a wide spread opinion, there should exist a glueball at $\simeq 1.5$ GeV [10–17]. This particle is not expected to be a pure state – it should be a mixture of the glueball and the isoscalar $q\bar{q}$ state of a nonet. That creates a decuplet. The abundance of the scalar mesons suggests that there exists more than one multiplet – we may expect a nonet and a decuplet. The problem is how the particles are distributed among them. A solution can be found if we exploit the accessible knowledge about these multiplets.

That can be achieved by means of the exotic commutator method (ECM) [18]. Using this method a system of algebraic equations (“master equations” (ME)) was derived for the octet contents of the isoscalar members of the zero-width meson nonet and decuplet [18, 19]. Solution of the ME gives full attainable information about these multiplets. For the nonets it clearly distinguishes three kinds of them: Gell-Mann–Okubo (GMO), Schwinger (S) and ideally mixed (I) ones. The differences matter in the analysis of the data. But ME gives not only the old-standing relations, as the mass formulae and an expression for the mixing angle of the nonet, but also something new (derived, as yet, only in the ECM approach): the nonet and decuplet mass ordering rules, the decuplet mass formula and the decuplet mixing matrix. The latter follows directly from the solution of the ME, without additional assumptions which are needed in other approaches for diagonalizing the unphysical mass operator. The mass ordering rules help in composing the multiplets of scalar mesons and make the description of the whole collection of the scalar mesons simple and transparent.

This method was also applied to describing a finite-width (complex-mass) mesons. Many nonets with different J^{PC} were fitted [9]. The fits demonstrate that most of the observed nonets are the S ones. Besides, extension of the ME to the complex mass reveals that the widths of the S nonet mesons depend linearly on the masses of the particles. The slope of the line is negative for all known nonets. The linearity follows from the flavor properties of the nonet. It is broken in all observed low mass nonets. The mechanism of the breaking is “kinematical” – it does not depend on the flavor coupling constant. This fact is important for the interpretation of the suppression of the $f_0(980)$ and $a_0(980)$ meson decays.

The part of the present paper concerning the nonet of 0^{++} mesons is a continuation of the previous analysis [9]. We justify the status and confirm the properties of the scalar nonet which were admitted to be still controversial there.

Our purpose is also to discuss the fit of the finite-width decuplet of the 0^{++} mesons, but first we have to make ME predictions for this multiplet. Therefore, we begin with recalling the ME procedure, fixing also the notation.

2 Exotic commutators and master equations for octet contents of the physical isoscalar states

The following sequence of exotic commutators is assumed to vanish [18]:

$$\left[T_a, \frac{d^j T_b}{dt^j} \right] = 0 \quad (j = 1, 2, 3, \dots), \quad (3)$$

where T is a $SU(3)_F$ generator, t is the time and (α, β) is an exotic combination of indices, i.e., such that the operator $[T_\alpha, T_\beta]$ does not belong to the octet representation. Substituting $\frac{dT}{dt} = i[H, T]$, and using the infinite momentum approximation for one-particle hamiltonian $H = \sqrt{m^2 + p^2}$, we transform (3) into the system

$$\begin{aligned} [T_\alpha, [\hat{m}^2, T_\beta]] &= 0, \\ [T_\alpha, [\hat{m}^2, [\hat{m}^2, T_\beta]]] &= 0, \\ [T_\alpha, [\hat{m}^2, [\hat{m}^2, [\hat{m}^2, T_\beta]]]] &= 0, \\ &\dots \end{aligned} \quad (4)$$

where \hat{m}^2 is the squared-mass operator.

For the matrix elements of the commutators (4) between one-particle states (we assume one-particle initial, final and intermediate states) we obtain the sequence of equations involving the expressions $\langle z_8 | (m^2)^j | z_8 \rangle$ with different powers $j = 1, 2, 3, \dots$ (z_8 is the isoscalar state belonging to the octet). Solving these equations, we obtain the sequence of formulae for a multiplet of the light mesons. We find

$$\langle z_8 | (\hat{m}_c^2)^j | z_8 \rangle = \frac{1}{3} a_c^j + \frac{2}{3} b_c^j \quad (j = 1, 2, 3, \dots). \quad (5)$$

Here \hat{m}_c^2 is assumed to be a complex-mass squared operator [9]:

$$\hat{m}_c^2 = \hat{m}^2 - i\hat{m}\hat{\Gamma}. \quad (6)$$

This operator can be diagonalized and has orthogonal eigenvectors. For the complex masses of the individual particles we use the following notation:

$$\begin{aligned} a_c &= a - i\alpha = (m_a)^2 - im_a\Gamma_a, \\ K_c &= K - i\kappa = (m_K)^2 - im_K\Gamma_K, \\ z_j &= x_j - iy_j = (m_j)^2 - im_j\Gamma_j, \\ z_8 &= x_8 - iy_8 = (m_8)^2 - im_8\Gamma_8, \\ b_c &= b - i\beta = (m_b)^2 - im_b\Gamma_b. \end{aligned} \quad (7)$$

The symbols a and K mean isotriplet and isodoublet meson respectively; z_j are isoscalar mesons; the real and imag-

inary parts of the subsidiary complex masses z_8 and b_c are

$$x_8 = \frac{1}{3}(4K - a), \quad y_8 = \frac{1}{3}(4\kappa - \alpha), \quad (8)$$

$$b = 2K - a, \quad \beta = 2\kappa - \alpha. \quad (9)$$

Numbering of the physical isoscalar mesons z_i is chosen such that their masses obey the inequality

$$x_i < x_{i+1}. \quad (10)$$

The octet state $|z_8\rangle$ can be expressed by the physical isosinglet states $|z_i\rangle$. For the nonet we substitute into (5) the expression

$$|z_8\rangle = l_1|z_1\rangle + l_2|z_2\rangle, \quad (11)$$

and for the decuplet

$$|z_8\rangle = l_1|z_1\rangle + l_2|z_2\rangle + l_3|z_3\rangle. \quad (12)$$

The coefficients l_i are complex numbers satisfying

$$\Sigma |l_i|^2 = 1. \quad (13)$$

As a result, we obtain the linear system of *master equations* (ME) determining the octet contents $|l_i|^2$ of the isoscalar z_i states:

$$\Sigma |l_i|^2 z_i^j = \frac{1}{3} a_c^j + \frac{2}{3} b_c^j \quad (j = 0, 1, 2, 3, \dots), \quad (14)$$

where the equation for $j = 0$ takes into account the condition (13). ME can be applied to analyzing the nonet and decuplet of the real and complex-mass mesons in the broken $SU(3)_F$ symmetry. The mass formulae arise if the number of equations exceeds the number of unknown coefficients l_i . They play the role of solvability condition of the system (14).

3 Nonet of 0^{++} mesons – back to $q\bar{q}$

3.1 Three kinds of the nonets

Solutions of the ME (14) for $|l_1|^2$, $|l_2|^2$ and the mass formulae for a nonet have already been analyzed [9]. We report on the main points of that analysis.

The system (14) can be solved if the number of equations ≥ 2 . In such cases $|l_1|^2$ and $|l_2|^2$ can be determined from the first two of them. We find

$$|l_1|^2 = \frac{1}{3} \frac{(z_2 - a_c) + 2(z_2 - b_c)}{z_2 - z_1}, \quad (15)$$

$$|l_2|^2 = \frac{1}{3} \frac{(a_c - z_1) + 2(b_c - z_1)}{z_2 - z_1}. \quad (16)$$

Then the subsequent equations have to be identically satisfied. These identities are complex-mass formulae. A different number of mass formulae define different kinds of the nonet.

No condition of solvability and, respectively, no mass formula exists for the system of the first two ME. This system can be written in the form

$$z_1 \sin^2 \theta + z_2 \cos^2 \theta = z_8, \quad (17)$$

where θ is mixing angle and z_8 is the Gell-Mann–Okubo mass squared:

$$z_8 \equiv \frac{1}{3} a_c + \frac{2}{3} b_c. \quad (18)$$

The formula (17) is sometimes considered as the nonet mass formula. Such a nonet is called the GMO one. It arises for the system of two equations (14).

For the system of three equations (14) we get one mass formula:

$$(a_c - z_1)(a_c - z_2) + 2(b_c - z_1)(b_c - z_2) = 0. \quad (19)$$

This is the Schwinger (S) complex-mass formula.

From the system of four equations (14), besides the mass formula (19), we obtain

$$a_c(a_c - z_1)(a_c - z_2) + 2b_c(b_c - z_1)(b_c - z_2) = 0. \quad (20)$$

From (19) and (20), choosing the numbers of z_i according to the rule (10), we get

$$z_1 = a_c, \quad z_2 = b_c, \quad |l_1|^2 = \frac{1}{3}, \quad |l_2|^2 = \frac{2}{3}. \quad (21)$$

This is a complex-mass ideally mixed nonet.

The system including one more equation (14) gives one more mass formula:

$$a_c^2(a_c - z_1)(a_c - z_2) + 2b_c^2(b_c - z_1)(b_c - z_2) = 0, \quad (22)$$

which is satisfied by the ideally mixed nonet. It is now obvious that also the subsequent equations of the system (14) comply with ideality [18].

We thus find that ECM predicts just three kinds of complex-mass nonets: GMO, S and I. Each of them defines three kinds of connections between real quantities:

1. between real parts of the complex-mass squared: the mass formulae;
2. between masses and widths of the particles: defining the flavor stitch line of the masses on the complex plane;
3. between imaginary parts of the complex-mass squared: width sum rules.

An important property of the mass formulae for the nonet of any kind is their independence of the particle widths and coincidence with respective mass formulae for the real-mass meson nonet. Therefore, the complex-mass meson nonets may be given the names of the real-mass ones: GMO, S and I [9]. Following the property of the independence of the mass formulae on the particle widths, also the definitions of these nonets are independent of them; the mesons forming different width patterns create the same nonet, if their masses are the same. The states of the nonet of any kind have the $q\bar{q}$ structure and this structure is

stable under anomalies of the widths. So, disagreement between expected and observed values of the $f_0(980)$ meson widths cannot be considered as evidence against the $q\bar{q}$ nature of this meson.

On the contrary, the $q\bar{q}$ structure of $f_0(980)$ meson is confirmed, if we indicate the nonet that it belongs to.

Experimental fits show that the well known meson nonets 1^{--} , 2^{++} and 3^{--} are the S ones. Also the less known 1^{+-} , 1^{++} and 2^{-+} are probably the S nonets. (Only the pseudoscalar mesons π , K , η , η' form the GMO nonet.) That, and the identity of the $a_0(980)$ and $f_0(980)$ meson masses, suggests that scalar mesons form the S nonet. We are looking for the nonet including $a_0(980)$, $K_0(1430)$ and $f_0(980)$ mesons. Then, the S mass formula indicates the $f_0(1710)$ meson as the ninth member.

We recall now the main properties of the mass formula and flavor stitch line for the S nonet. There exists one relation between the complex masses in this case.

3.2 Schwinger nonet mass formula for finite-width mesons

The S mass formula can be written in the form [9]

$$(a - x_1)(a - x_2) + 2(b - x_1)(b - x_2) = 0. \quad (23)$$

For the $|l|^2$ we find

$$|l_1|^2 = \frac{1}{3} \frac{(x_2 - a) + 2(x_2 - b)}{x_2 - x_1}, \quad (24)$$

$$|l_2|^2 = \frac{1}{3} \frac{(a - x_1) + 2(b - x_1)}{x_2 - x_1}. \quad (25)$$

They have to satisfy the condition $|l_i|^2 > 0$. This condition and the mass formula (23) make the particle masses comply with the ordering rule:

$$x_1 < a < x_2 < b, \quad (26)$$

or

$$a < x_1 < b < x_2. \quad (27)$$

The mixing angle Θ^{Sch} is real and totally determined by the masses:

$$\tan^2 \Theta^{\text{Sch}} = \frac{|l_1|^2}{|l_2|^2}. \quad (28)$$

This formula shows that also the mixing angle does not depend of the widths.

The masses fit the inequality $f(1710) < b$, pointing out the mass ordering (26) for this nonet. So the mass of the $f_0(980)$ meson must be smaller than the mass of the $a_0(980)$ one. The mass formula is well satisfied and the mixing angle ($\theta = 33.5 \pm 2$) $^\circ$ is close to ideal. For $f_0(1710)$ almost pure the $s\bar{s}$ structure is predicted.

Thus the scalar nonet looks quite ordinary. One may wonder, however, why it is so distinct from the other ones. This question can be explained to some extent by inspection of its flavor stitch line.

3.3 The flavor stitch line

The total widths of all physical mesons belonging to the S nonet, as well as the subsidiary states z_8 and b_c , satisfy the equation

$$\frac{\Gamma_k - \Gamma_l}{m_k - m_l} = k_s, \quad (29)$$

where k and l ($l \neq k$) run over a , K , x_1 , x_2 , x_8 , and b . Equation (29) represents a straight line in the (m, Γ) plane; k_s is its slope. Rectilinearity is an effect of requiring the ME (14) to hold for complex masses, i.e. that the relations between widths, and likewise between the masses, are completely determined by flavor interaction. The points (m, Γ) , representing the complex masses of the individual particles, form a sequence of stitches along the straight line. We call this the flavor stitch line (FSL).

Nonets of physical particles obey the equation of the flavor stitch line when sufficiently much decay channels are opened, so that the Γ are not sensitive to suppression of a single mode of decay. The data suggest that this happens as all masses of the nonet are bigger than ≈ 1.5 GeV. We refer to the width determined from FSL as the flavor

Table 1. The nonet of 0^{++} mesons. The three rows contain masses; widths; mixing angle, mass and width ordering. Subsidiary quantities $m_b = \sqrt{b}$ and $\Gamma_b = \frac{\beta}{m}$ are calculated. The large value of Γ_b reflects strong “kinematical” suppression of the a_0 decay. In the ordering rules a , b , x_1 , x_2 are masses squared; α , β , y_1 , y_2 are products of the mass and width. Masses and widths are given in MeV. Status of the particles, notation and data are quoted from RPP [2]

| J^{PC} | m_K | m_a | m_1 | m_b | m_2 |
|---------------|------------------------|---------------------|---------------|------------------------------|--------------|
| particles | Γ_K | Γ_a | Γ_1 | Γ_b | Γ_2 |
| | θ^{GMO} | mass ordering | | width ordering | |
| 0^{++} | 1412 ± 6 | 984.7 ± 1.2 | 980 ± 10 | 1737 ± 11 | 1714 ± 5 |
| • $a_0(980)$ | | | | | |
| • $K_0(1430)$ | 294 ± 23 | $50 \div 100$ | $40 \div 100$ | $380 \div 490$ | 140 ± 10 |
| • $f_0(980)$ | | | | | |
| • $f_0(1710)$ | $(33.5 \pm 2.0)^\circ$ | $x_1 < a < x_2 < b$ | | $y_1 > \alpha > y_2 > \beta$ | |

width (FW) of a meson. In the S nonet of the less massive particles, some of them may have the widths reduced by additional “kinematical” mechanism(s) suppressing decay. The outcome is the hadronic width (HW) – the quantity which is observed in experiment. The difference between FW and HW is a measure of the extra suppression of the decay. Disagreement between (29) and the data is best seen on the mass–width diagram. In this diagram FSL is a straight line. The point (m, Γ) of a physical state lying below FSL exhibits the particle having a reduced width due to a “kinematical” mechanism. Probably in most cases it is really kinematical (conservation laws, selection rules, phase space etc.), but there are also possible other (known or unknown) mechanisms of suppression; among them also the dynamical ones. In the same way the point (m, Γ) of the physical particle lying over FSL would exhibit the enhanced decay. Obviously, definition of the S nonet is invariant upon “kinematical” suppression.

The slope k_s of the FSL is not predicted by the model. It can be determined only with the help of the experimental data. From the nonets where the data are sufficient, we find

$$k_s = -0.5 \pm 0.1. \quad (30)$$

The value of k_s is firmly determined when all points (m, Γ) of the physical particles lie on a straight line. Also we can approximately (not so definite) determine it, using data on two particles, if we are convinced that their decays are not suppressed. One of the reasons for such a conviction is $m > 1.5$ GeV.

The mass–width diagram of the 0^{++} nonet is shown in Fig. 1. An approximate FSL is determined by the (m, Γ) coordinates of the $K_0(1430)$ and $f_0(1710)$ mesons. Its slope, $k_s = -0.56$, is typical for the S nonets (30). The FW of $f_0(980)$ meson, $\Gamma \approx 535$ MeV, is consistent with the once predicted width $\Gamma^{q\bar{q}}$ of the $f_0(980)$ meson, (2).

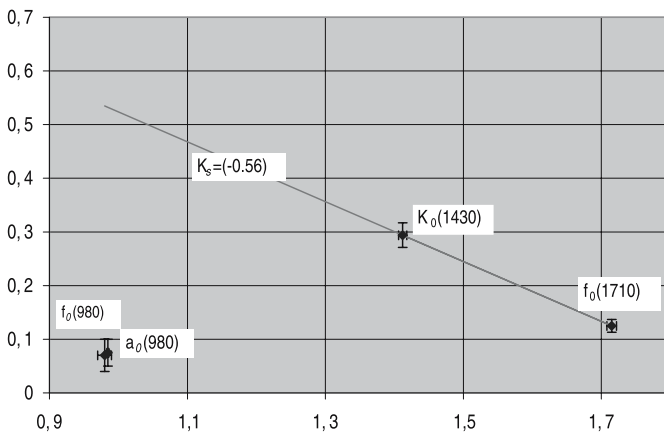


Fig. 1. Mass–width diagram of the 0^{++} nonet. On the axes m and Γ are given in GeV. The approximate flavor stitch line is drawn according to the coordinates of the $K_0(1430)$ and $f_0(1710)$ mesons. The large deficit of the $f_0(980)$ and $a_0(980)$ widths demonstrates strong “kinematical” suppression of their decays. Equal rates of the suppression emphasize its flavor independence

According to Fig. 1, the FW of the $a_0(980)$ meson is the same. The observed HW are $40 \div 100$ MeV for $f_0(980)$, and $50 \div 100$ MeV for $a_0(980)$; so their HW and suppression rates are also the same. Equality of the suppression rates of the isovector and isoscalar meson decays is quite exceptional in the low lying nonets. In other nonets (except of the “kinematically” unsuppressed 3^{--}) these rates are different. This suggests that for both, the $a_0(980)$ and the $f_0(980)$ mesons works the same suppression mechanism and so it is isospin independent. Another feature of this mechanism is that it does not change the masses, but this only confirms the “kinematical” nature of the suppression.

So what can be the physical nature of the suppression? Our analysis does not answer this question. Perhaps we should turn to those effects which, according to current opinion, can modify properties of the scalar mesons like confinement, vacuum effects, violation of the chiral symmetry, ...

4 Decuplet of mesons. Glueball mixing

4.1 The masses, widths and flavor stitch line

The decuplet of the meson states is a reducible representation of $SU(3)_F$ symmetry:

$$10 = 8 \oplus 1 \oplus 1. \quad (31)$$

It arises by joining an additional singlet with a nonet. Below, the singlet is considered as a glueball and the nonet as a $q\bar{q}$ system, but it is not necessary to specify them. We call the latter a “basic nonet” of the decuplet.

ECM gives the unique possibility of a simple and transparent description of the multiplets of complex-mass mesons. The description of a decuplet is, in essence, identical with the description of the nonet and is based on the same ME (c.f. (5)), but with the octet state $|z_8\rangle$ given by (12). We thus have to solve the system of linear equations

$$|l_1|^2 z_1^j + |l_2|^2 z_2^j + |l_3|^2 z_3^j = \frac{1}{3} a_c^j + \frac{2}{3} b_c^j \quad (j = 0, 1, 2, \dots), \quad (32)$$

with respect to the $|l_i|^2$ (c.f. (14)). The solution can exist if the number of equations is three or more. We postulate four equations, i.e. the vanishing of three exotic commutators. A nonet satisfying this system of exotic commutators is ideal; in the decuplet, the ideal structure of the basic nonet states is violated due to mixing with the glueball. We say that the basic nonet of the decuplet is ideal.

For the system of four equations (32) we have one solvability condition relating the complex masses of the decuplet. It will be seen further that this relation, along with the requirement of positivity of the $|l_i|^2$, leads to the mass ordering rule as another necessary condition of solvability. The ordering rule and the mass formula help much in

completing the decuplet, making the procedure simple and transparent.

The solution of (32) is

$$|l_1|^2 = \frac{1}{3} \frac{(z_2 - a_c)(z_3 - a_c) + 2(z_2 - b_c)(z_3 - b_c)}{(z_1 - z_2)(z_1 - z_3)}, \quad (33)$$

$$|l_2|^2 = \frac{1}{3} \frac{(z_1 - a_c)(z_3 - a_c) + 2(z_1 - b_c)(z_3 - b_c)}{(z_2 - z_1)(z_2 - z_3)}, \quad (34)$$

$$|l_3|^2 = \frac{1}{3} \frac{(z_1 - a_c)(z_2 - a_c) + 2(z_1 - b_c)(z_2 - b_c)}{(z_3 - z_1)(z_3 - z_2)}, \quad (35)$$

provided the complex masses of the particles satisfy the equation

$$M \stackrel{def}{=} (z_1 - a_c)(z_2 - a_c)(z_3 - a_c) + 2(z_1 - b_c)(z_2 - b_c)(z_3 - b_c) = 0. \quad (36)$$

The $|l_i|^2$ must be real numbers. It can easily be seen that all $\text{Im}|l_i|^2 = 0$ if the equations (c.f. (29))

$$\frac{y_i - y_j}{x_i - x_j} = \frac{y_i - \alpha}{x_i - a} = \frac{y_i - \beta}{x_i - b} = \frac{\alpha - \beta}{a - b} = k_s \quad (37)$$

are satisfied for all i, j ($j \neq i$) running over z_1, z_2, z_3 and z_8 .

These equations define the FSL of the complex-mass meson decuplet. The points $(m^2, m\Gamma)$ of all mesons belonging to the decuplet, as well as of the subsidiary states z_8 and b_c , lie on the straight line with slope k_s in the $(m^2, m\Gamma)$ plane. Obviously, the points (m, Γ) corresponding to these mesons lie in the plane (m, Γ) on the straight line with the same slope.

The last equation, (37), shows that k_s can be defined by the parameters of the a and K mesons: the slope of the decuplet FSL is identical with the slope of the basic nonet one; joining the additional singlet does not change the slope of FSL, nor the stitch line. Also the other properties of the decuplet and the basic nonet are the same:

1. linearity follows from flavor symmetry and departures from it are a result of ‘‘kinematical’’ suppression (enhancement) of the decay, and
2. the slope k_s is not predicted and can only be determined by the data.

Using (37) we transform (33)–(35) into

$$|l_1|^2 = \frac{1}{3} \frac{(x_2 - a)(x_3 - a) + 2(x_2 - b)(x_3 - b)}{(x_1 - x_2)(x_1 - x_3)}, \quad (38)$$

$$|l_2|^2 = \frac{1}{3} \frac{(x_1 - a)(x_3 - a) + 2(x_1 - b)(x_3 - b)}{(x_2 - x_1)(x_2 - x_3)}, \quad (39)$$

$$|l_3|^2 = \frac{1}{3} \frac{(x_1 - a)(x_2 - a) + 2(x_1 - b)(x_2 - b)}{(x_3 - x_1)(x_3 - x_2)}. \quad (40)$$

They coincide with the $|l_i|^2$ for the zero-width meson decuplet [19–21].

Let us define now the two real functions

$$M_R \stackrel{def}{=} (x_1 - a)(x_2 - a)(x_3 - a) + 2(x_1 - b)(x_2 - b)(x_3 - b), \quad (41)$$

$$M_I \stackrel{def}{=} (y_1 - \alpha)(y_2 - \alpha)(y_3 - \alpha) + 2(y_1 - \beta)(y_2 - \beta)(y_3 - \beta). \quad (42)$$

From (37) it follows that

$$M_I = k_s^3 M_R. \quad (43)$$

Due to (37), the real and imaginary parts of the solvability condition (36) can be written in the form

$$\text{Re}M = M_R(1 - 3k_s^2) = 0, \quad (44)$$

$$\text{Im}M = M_I(1 - \frac{3}{k_s^2}) = 0. \quad (45)$$

If $k_s^2 \neq \frac{1}{3}$, then $M_R = 0$. The same is true also for $k_s^2 = \frac{1}{3}$, as can be seen from (45) and (43). Returning to the definition of M_R , (41), we find an explicit form of the decuplet mass formula for finite-width mesons:

$$(x_1 - a)(x_2 - a)(x_3 - a) + 2(x_1 - b)(x_2 - b)(x_3 - b) = 0. \quad (46)$$

It does not depend on the particle widths and is identical with the mass formula for the zero-width mesons [20].

The equation $M_I = 0$ determines a sum rule for the decuplet particle widths:

$$(y_1 - \alpha)(y_2 - \alpha)(y_3 - \alpha) + 2(y_1 - \beta)(y_2 - \beta)(y_3 - \beta) = 0. \quad (47)$$

This equation is satisfied only if all points (m, Γ) lie on the FSL.

The right-hand parts of (38)–(40) must be positive. This cannot be fulfilled for arbitrary masses. Therefore, the requirement of positivity restricts the masses. These restrictions along with the mass formula lead to the rule of the mass ordering as the necessary condition of solvability of the system (32) [19–21]:

$$x_1 < a < x_2 < b < x_3. \quad (48)$$

For the imaginary parts of the complex masses we have

$$y_1 < \alpha < y_2 < \beta < y_3. \quad (49)$$

The basic nonet of the decuplet could also be chosen as an S one. The S nonet follows from the assumption that two exotic commutators (3) vanish. Then we would have three ME, just enough for determining the $|l_i|^2$. As we know from fits of the meson nonets [9], the S nonet is not very different from the I one and we hope that the properties of the decuplets built on them are not very different. However, choosing the S basic nonet we would be left without the mass formula and the ordering rule. Therefore, we do not discuss this scheme.

To finish this section, let us note that there are no more types of decuplet in the ME approach. For five ME (32) there arise two complex-mass formulae; beside of (46), we obtain

$$a_c(z_1 - a_c)(z_2 - a_c)(z_3 - a_c) + 2b_c(z_1 - b_c)(z_2 - b_c)(z_3 - b_c) = 0. \quad (50)$$

These formulae define the ideal nonet and disconnected unitary singlet with arbitrary mass. This result does not change if we join the next ME, (32).

4.2 Completing the decuplet

Completing the decuplet is quite easy due to the mass ordering rule and the simplicity of the mass formula.

The decuplet should include the isoscalar components not belonging to the nonet: $f_0(1370)$ ($\equiv z_1$) and $f_0(1500)$ ($\equiv z_2$). The latter one is considered by many authors [12–17] as the most likely glueball candidate. In the same mass region we find the isotriplet meson $a_0(1450)$. If these three mesons belong to the same multiplet, irrespectively to the nonet or to the decuplet, then, according to the mass ordering rule, their masses should obey the inequalities

$$x_1 < a < x_2. \quad (51)$$

They could belong to the nonet, if there existed a K_0 meson with such a mass that,

$$b = 2K_0 - a_0(1450), \quad (52)$$

(c.f. (9)) not much exceeds x_2 . Then the meson $f_0(1500)$ would have a structure close to $s\bar{s}$. Such a K_0 meson is not observed. On the other hand, nobody expects $f_0(1500)$ to have the $s\bar{s}$ structure. Therefore, the needed meson K_0 should have a higher mass and the multiplet should be a decuplet.

The only K_0 candidate which may play this role is $K_0(1950)$, still “needing confirmation”. If it is accepted, then only one isoscalar meson f_0 is lacking to complete the decuplet. At present, several signals are announced [2]:

$$f_0(2020), f_0(2060), f_0(2100), f_0(2200), f_0(2330). \quad (53)$$

The tenth candidate should be distinguished by the value of the mass calculated from the mass formula (46). However, on account of the large error of most of the input masses, we first perform a qualitative discussion of the formula.

An exceptionally large difference between the masses of the $a_0(1450)$ and $K_0(1950)$ mesons, $m_K - m_a \simeq 500$ MeV, enables us to estimate very precisely the difference $x_3 - b$. (Notice, by the way, that such a large difference between appropriate masses is observed also in the nonet 0^{++} .) According to the mass ordering rule, this difference must be positive. For such a large mass of K_0 the difference $b - a$ is also large ($b - a \simeq 5.5$ GeV²) and we have

$$x_3 - a \approx b - a, \quad b - x_1 \approx b - x_2 \approx b - a. \quad (54)$$

Then, from (46) we find

$$x_3 = b + \frac{(x_2 - a)(a - x_1)}{2(b - a)}. \quad (55)$$

Following the opinion of [12–17] that the glueball dominates the structure of the z_2 ($\equiv f_0(1500)$) meson, we admit that the light quarks dominate the structure of the z_1 ($\equiv f_0(1370)$) meson; the mass of z_1 should be closer to the mass of a_0 than the mass of z_2 and the masses of a_0 , z_1 , z_2 mesons would satisfy the inequality

$$a - x_1 < x_2 - a. \quad (56)$$

So we find

$$b < x_3 < b + \frac{(x_2 - a)^2}{2(b - a)}, \quad (57)$$

or

$$x_3 - b < 1MeV^2. \quad (58)$$

The poorly known masses of $K_0(1950)$ and f_0 mesons appear to be strongly correlated.

The particles creating the decuplet are mentioned in Table 2. The last column shows the mass of the b -state m_b . The value of this mass singles out $f_0(2330)$ as the best candidate for the heaviest isoscalar of the decuplet.

The mass of the b -state shown in Table 2 has a large uncertainty due to the error of the $a_0(1450)$ mass and, especially, to the uncertainty of the $K_0(1950)$ mass. Therefore, it may be interesting to admit another particle – the $f_0(2200)$ – as the third isoscalar component and construct an adequate decuplet. We can do this, keeping $f_0(1500)$, the mostly glueball state with unchanged mass (the best known mass in the decuplet) and allowing to change the other masses as to obey the mass formula (46).

Table 3 shows these two possible decuplets. Besides the masses of the particles also are shown the octet contents of the isoscalar mesons. A common feature of these two solutions of the ME is the small value of $|l_2|^2$. This follows from the assumption that $f_0(1500)$ meson is mostly the glueball state. The mixing matrices belonging to these solutions are presented in the next section and denoted by V_1 and V_2 .

4.3 Mixing matrix

Joining the glueball state with the quark nonet states raises the problem of constructing the mixing matrix for three isoscalar states. This problem was so far formulated only for zero-width mesons. In this case, the mixing matrix is obtained by diagonalizing some postulated unphysical mass operator. The most general form of this operator is simplified by additional assumptions which reduce the number of independent parameters, to facilitate the diagonalization [22]. That makes the result model dependent. It would be still more difficult to obtain any result for the finite-width meson decuplet in this way.

ECM provides another procedure of constructing the mixing matrix. It is based on the solution of the ME (14) given by $|l_1|^2$, $|l_2|^2$, $|l_3|^2$, see (38)–(40), defining the octet

Table 2. The decuplet of scalar mesons. The decuplet is formed out of the mesons satisfying the decuplet mass formula (46). Three of them are well known. The remaining two (K_0 and $f_0(2330)$) have not firmly been established. Their masses are strongly correlated which supports them mutually as candidates to the decuplet. The predicted ordering rules for masses and widths are given in the last row. The width ordering rule cannot be verified with the present data. For notation, see the caption of the Table 1

| 0^{++} particles | m_K Γ_K | m_a Γ_a | m_1 Γ_1 | m_2 Γ_2 | m_3 Γ_3 | m_b Γ_b |
|---|---------------------------|---------------------|---------------------|------------------------------------|---------------------|---------------------|
| | mass ordering | | | width ordering | | |
| • $a_0(1450)$ $K_0(1950)$ | 1945 ± 30 | 1474 ± 19 | $1200 \div 1500$ | 1507 ± 5 | 2330 | 2325 ± 92 |
| • $f_0(1370)$ • $f_0(1507)$ $f_0(2330)$ | 201 ± 113 | 265 ± 13 | $200 \div 500$ | 109 ± 7 | 220 | |
| | $x_1 < a < x_2 < b < x_3$ | | | $y_1 > \alpha > y_2 > \beta > y_3$ | | |

Table 3. Two possible solutions of the ME (46) for decuplet adequate to the choice of the $f_0(2330)$ and $f_0(2200)$ meson as the heaviest isoscalar state. In both cases the solution is chosen such that $f_0(1500)$ is mostly glueball. One can see that the small content of the octet state is the signature of a glueball. Masses are given in MeV. Notation and data are quoted from RPP [2]

| Solution number | m_K | m_a | m_1 $ l_1 ^2$ | m_2 $ l_2 ^2$ | m_3 $ l_3 ^2$ |
|--------------------|--------|--------|--------------------|--------------------|--------------------|
| 1 | 1945.0 | 1474.0 | 1465.0 0.25668 | 1505.98 0.07691 | 2322.45 0.66642 |
| 2 | 1870.0 | 1460.0 | 1443.0 0.23841 | 1507.66 0.09590 | 2205.34 0.66569 |

contents of the isoscalar mesons [19, 21]. There is no need for introducing the mass operator nor making assumptions about the mixing mechanism, except the natural conjecture about the flavor independence of the glueball. The octet contents are expressed by physical masses and nothing else. The method enables one to construct equally easy the mixing matrix both for zero-width particles and finite-width ones. We will calculate it in the latter case.

Let us introduce the mixing matrix U transforming isoscalar states of exact $SU(3)_F$ symmetry into the physical ones:

$$\begin{bmatrix} |z_1\rangle \\ |z_2\rangle \\ |z_3\rangle \end{bmatrix} = U \begin{bmatrix} |z_8\rangle \\ |z_0\rangle \\ |G\rangle \end{bmatrix}, \quad (59)$$

where z_0 is a $q\bar{q}$ singlet and G is a glueball. For complex-mass particles the matrix is, in general, unitary:

$$U = \begin{bmatrix} c_1 & -s_1c_2 & s_1s_2 \\ s_1c_3 & c_1c_2c_3 - s_2s_3e^{i\delta} & -c_1s_2c_3 - c_2s_3e^{i\delta} \\ s_1s_3 & c_1c_2s_3 + s_2c_3e^{i\delta} & -c_1s_2s_3 + c_2c_3e^{i\delta} \end{bmatrix}. \quad (60)$$

Here $c_j = \cos \vartheta_j$; $s_j = \sin \vartheta_j$; $j = 1, 2, 3$; ϑ_j are the Euler angles: $0 \leq \vartheta_1 < \pi$; $0 \leq \vartheta_2, \vartheta_3 < 2\pi$; δ is an arbitrary phase. The elements of the first column are l_1, l_2, l_3 , i.e., the coefficients which were introduced in (12). The squared absolute values of these coefficients are the solutions (38)–(40) of the system (32). Therefore, we have

$$c_1 = \pm \sqrt{|l_1|^2}; \quad s_1c_3 = \pm \sqrt{|l_2|^2}; \quad s_1s_3 = \pm \sqrt{|l_3|^2}. \quad (61)$$

Thus the angles ϑ_1 and ϑ_3 are determined by the masses up to the signs of c_1, c_3, s_3 .

To compare the predictions with the data, the mixing matrix is usually expressed in the basis of the ideal nonet states

$$|N\rangle = \frac{1}{\sqrt{2}}|(u\bar{u} + d\bar{d})\rangle, \quad |S\rangle = |s\bar{s}\rangle. \quad (62)$$

The physical isoscalar states are

$$\begin{bmatrix} |z_1\rangle \\ |z_2\rangle \\ |z_3\rangle \end{bmatrix} = V \begin{bmatrix} |N\rangle \\ |S\rangle \\ |G\rangle \end{bmatrix}, \quad (63)$$

where

$$V = UQ, \quad (64)$$

and the matrix Q ,

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} & 0 \\ \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (65)$$

transforms the two bases

$$\begin{bmatrix} |z_8\rangle \\ |z_0\rangle \\ |G\rangle \end{bmatrix} = Q \begin{bmatrix} |N\rangle \\ |S\rangle \\ |G\rangle \end{bmatrix}. \quad (66)$$

The explicit form of the matrix V is

$$V = \begin{bmatrix} \frac{1}{\sqrt{3}}c_1 - \sqrt{\frac{2}{3}}s_1c_2 & -\sqrt{\frac{2}{3}}c_1 - \frac{1}{\sqrt{3}}s_1c_2 & s_1s_2 \\ \frac{1}{\sqrt{3}}s_1c_3 + \sqrt{\frac{2}{3}}(c_1c_2c_3 - s_2s_3e^{i\delta}) & -\sqrt{\frac{2}{3}}s_1c_3 + \frac{1}{\sqrt{3}}(c_1c_2c_3 - s_2s_3e^{i\delta}) & -c_1s_2c_3 - c_2s_3e^{i\delta} \\ \frac{1}{\sqrt{3}}s_1s_3 + \sqrt{\frac{2}{3}}(c_1c_2s_3 + s_2c_3e^{i\delta}) & -\sqrt{\frac{2}{3}}s_1s_3 + \frac{1}{\sqrt{3}}(c_1c_2s_3 + s_2c_3e^{i\delta}) & -c_1s_2s_3 + c_2c_3e^{i\delta} \end{bmatrix} \quad (67)$$

The angle ϑ_2 can also be determined if flavor independence of the glueball is imposed:

$$\langle u\bar{u}|m^2|G\rangle = \langle d\bar{d}|m^2|G\rangle = \langle s\bar{s}|m^2|G\rangle. \quad (68)$$

For the state $|z_8\rangle$ this reads

$$\langle z_8|m^2|G\rangle = 0. \quad (69)$$

Substituting here $|z_8\rangle$ and $|G\rangle$, expressed from (59) and (60), we find the following equation for ϑ_2 :

$$\tan \vartheta_2 = e^{-i\delta} \frac{c_3s_3}{c_1} \frac{z_3 - z_2}{(z_3 - z_1) - (z_3 - z_2)c_3^2}. \quad (70)$$

Separation of the real and imaginary parts of z_i leads, after obvious modifications, to the same equation, with z_i replaced by x_i . In this equation, $e^{-i\delta}$ is the only complex number. Therefore, $\delta = 0$ and ϑ_2 is completely determined by the masses:

$$\tan \vartheta_2 = \frac{c_3s_3}{c_1} \frac{x_3 - x_2}{(x_3 - x_1) - (x_3 - x_2)c_3^2}. \quad (71)$$

V is now orthogonal matrix. It is independent of the particle widths, has no free parameters and is identical to the mixing matrix of the isoscalar zero-width particle states [19, 21].

Thus only signs of the trigonometric functions c_j, s_j remain to determine. We can choose them in the following way. Three elements of the mixing matrix, V_{11}, V_{13}, V_{32} , may be chosen positive. Then,

- $s_1 > 0$, since $0 \leq \vartheta_1 < \pi$;
- $c_1 > 0, c_2 < 0$, if we expect $|z_1\rangle \approx |N\rangle$ with $V_{11} > 0$;
- $s_2 > 0$, as $V_{13} > 0$;
- $s_3 < 0, c_3 > 0$, if we expect $|z_2\rangle \approx |S\rangle$ with $V_{32} > 0$.

This choice of signs is consistent with (71).

The mixing matrices for the solutions mentioned in Table 3 read

$$V_1 = \begin{bmatrix} 0.88472 & 0.00510 & 0.46609 \\ 0.46612 & -0.01005 & -0.88466 \\ 0.00018 & 0.99994 & -0.01127 \end{bmatrix}, \quad (72)$$

$$V_2 = \begin{bmatrix} 0.86100 & 0.01081 & 0.50849 \\ 0.50861 & -0.01964 & -0.86077 \\ 0.00069 & 0.99975 & -0.02241 \end{bmatrix}. \quad (73)$$

4.4 Properties of the solutions

Solution of the ME for a decuplet consists of a mass formula connecting five masses and three expressions for $|l_i|^2$'s which determine the octet contents of the three isoscalar components. The latter are used for constructing the mixing matrix. This approach already has been applied for investigating the decuplets of the zero-width mesons [19–21, 23–26]. It remains unchanged for the finite-widths mesons.

Specifically for the 0^{++} decuplet, the properties of the solution are dominated by the large difference between the masses of the $a_0(1450)$ and $K_0(1950)$ mesons. This implies $x_3 \simeq b$ and enables us to make a motivated choice of f_0 from among (53). The Table 3 and the matrices V_1 and V_2 display two solutions of the ME corresponding to different f_0 . The solutions confirm the connection between the range of indefiniteness of the mass of $K_0(1950)$ and the mass range $2200 \div 2400$ of f_0 . The predicted properties of the solutions are similar, because the input masses of f_0 differ from one another much less than the masses of the $a_0(1450)$ and $K_0(1950)$ mesons. Table 2 and the matrices V_1 and V_2 also show that it would be difficult to make a choice between the signals (53), based only on the properties of the mixing matrix.

Another consequence of the large difference between the masses of $a_0(1450)$ and $K_0(1950)$ is that the third f_0 is a pure $s\bar{s}$ -state. Therefore, the G -state is included only into the $f_0(1370)$ - and $f_0(1500)$ -states. Its distribution is determined by the relations between the masses of the three mesons: $f_0(1370)$, $a_0(1450)$ and $f_0(1500)$. Precise knowledge of these masses is sufficient for a complete determination of the mixing matrix. As the present data are not accurate enough, we are guided in constructing the matrix V_1 and V_2 by the qualitative suggestion that G is contained mainly in the state of $f_0(1500)$. However, with the present data it is also possible, with a suitable choice of the masses, that G is contained mainly in the $f_0(1370)$ -state.

5 Summary and discussion

In this paper we discuss only the flavor properties of the scalar mesons imposed by broken $SU(3)_F$ symmetry. We extract as many predictions from this symmetry as possible and verify their consistency with the data. We neither consider the quark dynamics, nor the structure of the particles; in particular, we do not discuss the problem of

the structure of the higher lying multiplet (decuplet) (are they excited $q\bar{q}$ -states, hybrid $q\bar{q}g$ -states or something else [3, 27]?). Our approach does not require information about the structure of the components of the multiplet, but we hope that it may help in determining them.

The predictions of the flavor symmetry can be obtained by means of the exotic commutator method (ECM) of breaking the unitary symmetry. The requirement of disappearing of the matrix elements of these commutators between one-particle octet states gives the system of master equations (ME) which determine the octet contents of the isoscalar physical states. The ME include all information attainable for the multiplets of 0^{++} mesons. By solving them, we obtain not only all relations for these multiplets that are already known, but also the new ones. In particular, we get the relations for the nonet and decuplet of the finite-width mesons which were unknown before. The only parameters of the ME are physical masses and widths of the mesons. Therefore, the predictions do not depend on free parameters, additional assumptions or unphysical quantities. ECM provides the unique possibility for investigating the implications of the flavor symmetry in its pure and separated form.

ECM distinguishes three types of zero-width nonets: Gell-Mann–Okubo (GMO), Schwinger (S) and ideal (I) ones. They are defined as satisfying the corresponding mass formulae. For the finite-width nonets the mass formulae are also independent of the widths of the particles and are identical with those for the zero-width mesons. Therefore, we may keep the same definitions and names for them. Data show that all (with one exception) known nonets are of the type S. We consider this observation as an experimental fact and use it for choosing the candidate to the scalar nonet.

The mesons $a_0(980)$, $f_0(980)$ and $K_0(1430)$ are natural candidates to the nonet. The S mass formula singles out the $f_0(1710)$ as the ninth member of the nonet. This does not contradict the known properties of this particle, because it is recognized as an $s\bar{s}$ -state [17]. So these particles form a usual S nonet. Such an assignment cannot be affected by the width anomalies of the $f_0(980)$ and $a_0(980)$ mesons, because the definition of the nonet does not depend on the widths. Therefore, these anomalies cannot serve as an argument for introducing the exotic multiplet but should be explained on the basis of the $q\bar{q}$ structure.

The flavor symmetry predicts that all particles belonging to the S nonet form the straight flavor stitch line in the (m, Γ) plane. The data show that the slope of the stitch line is negative. The linearity may, however, be broken (for some particles having masses smaller than $\simeq 1.5$ GeV), if the usual flavor-conditioned decay is distorted by some “kinematical” mechanism. Such a kind of mechanism suppresses the decays of the $f_0(980)$ and $a_0(980)$ mesons. We know some of its properties: the suppression is strong; it does not depend on the masses; it is independent of the isospin. But the present approach does not identify its nature. The riddle of the scalar mesons remains.

Obviously, with this “kinematical” mechanism suppressing the $f_0(980)$ and $a_0(980)$ decays, the investigation of the $\delta_{J=0}^{I=0}$ and $\delta_{J=0}^{I=1}$ phases in the resonance region does

not yield information about the properties of the flavor symmetry.

Let us now discuss the decuplet. The decuplet is a real object – as real as the nonet. It is a multiplet whose isoscalar octet state is distributed among three isoscalar physical states. Its properties, as well as the properties of the nonet, are defined by ME. The number of equations may be chosen in such a way that the masses satisfy the mass formula. With such a choice, the mass formula plays the role of a necessary condition of the solvability of the ME. The formula does not depend on the widths of the particles. The solution of the ME, $|l_i|^2$ ($i = 1, 2, 3$), serves for constructing the mixing matrix of the decuplet. The matrix is based exclusively on the solution and is real for real masses as well as for complex ones. Its elements depend only on the masses. The particles of the decuplet states form the straight flavor stitch line on the (m, Γ) plane, just as do the S nonet ones.

The decuplet includes the mesons $a_0(1450)$, $K_0(1950)$, $f_0(1370)$ and $f_0(1500)$. The missing f_0 has a mass somewhere in the region $2200 \div 2400$ MeV. The mass is strongly correlated with the mass of $K_0(1950)$. Its vagueness reflects the inaccuracy of the $K(1950)$ mass and the error of the $a_0(1450)$ one. This meson is almost a pure $s\bar{s}$ -state; therefore the state G is almost completely distributed between the $f_0(1370)$ and $f_0(1500)$ mesons. The G content of each of these particles is determined by relations between the $f_0(1370)$, $a_0(1450)$ and $f_0(1500)$ masses. Thus the knowledge of these three masses is sufficient for approximately evaluating the whole mixing matrix.

The fit of the flavor stitch line should be a necessary element of the present investigation. One can expect a good fit, as the masses of the decuplet particles are large enough. The extra information about the widths would be especially welcome for the $f_0(1370)$ and $f_0(1500)$ mesons, which are expected to include a glueball. However, the current data are too crude for that. More data and a better understanding of the decay processes are desirable [28].

In general, consistency of the mixing parameters predicted from the decuplet masses with the results of the analysis of the isoscalar mesons production and decay would provide the requested ultimate evidence of G .

The mass regions of the nonet and decuplet are overlapping. However, such a situation should not be treated as an obstacle for accepting the proposed distribution of the particles between the multiplets. We may prefer to follow the data rather than the habitual mixing the adjacent states. The 0^{++} nonet mesons well satisfy the S mass formula. This suggests that mixing between the nonet and decuplet states is negligible. A similar situation can be seen for the 1^{--} multiplets: the ground state nonet (ρ , K^* , ω , ϕ) is ideally mixed, while the higher lying states ($\rho(1450)$, $K^*(1410)$, $\phi(1420)$) form the octet of exact symmetry [9]. That could not happen if there were a remarkable mixing between these multiplets. We can also notice that for many J^{PC} not only the ground state multiplet is observed, but also the higher one; it would be impossible to distinguish separate multiplets if the mixing were strong.

For the S nonet and decuplet, apart from the mass formulae, there exist other necessary solvability conditions of

the ME. They have the form of the mass ordering rule. For the 0^{++} nonet we have

$$x_1 < a < x_2 < b,$$

while for the 0^{++} decuplet it is

$$x_1 < a < x_2 < b < x_3.$$

These rules are very useful in investigating the nonet and decuplet of the scalar multiplets.

They throw also some light on the problem of the $\sigma(600)$ meson. According to these rules, the nonet of the 0^{++} mesons cannot be transmuted into a decuplet by joining the scalar meson with a mass smaller than the one of the $f_0(980)$. Therefore, $\sigma(600)$ cannot be considered as a decuplet component and is a separate state. It may be a genuine particle state (then it would be the ground state unitary singlet), or it may be a state of a different kind. A possibility that the nature of the $\sigma(600)$ is different from the nature of the other scalar mesons has been discussed for some time [29–31].

6 Conclusion

1. The most complete description of the meson multiplets (nonet and decuplet) is given by the master equations (ME) which are derived from the hypothesis of vanishing of the exotic commutators. For the finite-width mesons they reveal the features which were not known before. These features enable us to understand the mass spectrum of the scalar mesons.
2. The 0^{++} mesons form the nonet ($a_0(980)$, $K_0(1430)$, $f_0(980)$, $f_0(1710)$), the decuplet ($a_0(1450)$, $K_0(1950)$, $f_0(1370)$, $f_0(1500)$, $f_0(2200 \div 2400)$) and a separate state $\sigma(600)$.
3. There are no $q^2\bar{q}^2$ exotics. The nonet mesons satisfy the Schwinger mass formula and are the usual $q\bar{q}$ -states. Anomalies of the $f_0(980)$ and $a_0(980)$ widths are caused by some “kinematical” mechanism which suppresses their decay. The energy dependence of the phases $\delta_{J=0}^{I=0}$ and $\delta_{J=0}^{I=1}$ do not reflect properties of the flavor interaction. The nature of the suppression mechanism remains unknown.
4. The decuplet includes the glueball state. The mass formula and mixing matrix of the decuplet isoscalar physical states follow directly from the solution of the ME. The glueball is included in the states of the $f_0(1370)$ and $f_0(1500)$ mesons. Its contribution to these states is completely determined by the masses of the decuplet particles. Agreement between the quark–glueball structures of the isoscalar physical states, determined in this way, and their production and decay patterns would provide the ultimate identification of the glueball.
5. The meson $\sigma(600)$ cannot be mixed with the nonet ($a_0(980)$, $K_0(1430)$, $f_0(980)$, $f_0(1710)$) to form a decuplet. This may support the conjecture about the peculiar nature of this particle.

Acknowledgements. It is a pleasure for me to thank Profs. S.B. Gerasimov, P. Kosinski and V.A. Meshcheryakov for valuable discussions and Profs. P. Maslanka and J. Rembielinski for support during completing this work.

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